STATISTICAL INFERENCE FOR GENERALIZED LORENZ DOMINANCE BASED ON GROUPED DATA: A RECONSIDERATION

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First draft: July 2001
This version: September 17, 2002

ABSTRACT

One income distribution is preferable to another under any increasing and Schur-concave (S-concave) social welfare function if and only if the generalized Lorenz (GL) curve of the first distribution lies above that of the second (GL dominance). This paper (i) derives the asymptotic distribution of a vector of sample GL curve ordinates, interpreting it as a method-of-moments estimator, and (ii) proposes a simple simulation-based test for GL dominance (and for multiple inequality restrictions in general) that is consistent and asymptotically has the correct size. The paper also provides detailed Monte Carlo experiments and an application to income distributions in Japan.

Key Words: Generalized Lorenz Curve; Stochastic Dominance; Method of Moments; Multiple Inequality Restrictions

JEL Classification: C12, D31, D63.

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Contents

1 INTRODUCTION

While many empirical works provide estimates of various inequality measures to compare income or wealth inequality across regions or over time, they rarely report the standard errors; thus the readers rarely know about possible sampling errors in the estimates. Although statistical inference procedures do exist in the literature, few empirical works use them. This is perhaps because those procedures are somewhat complicated.

Among various criteria for comparing income (or wealth) distributions, this paper focuses on the generalized Lorenz (GL) curve for two reasons. First, Shorrocks (1983) shows that one income distribution is preferable to another under any increasing and Schur-concave (S-concave) social welfare function if and only if the GL curve of the first distribution lies above that of the second (GL dominance); hence the GL curve itself is an interesting object to study. Second, once we obtain the asymptotic distribution of a vector of sample GL curve ordinates, the delta method gives the asymptotic distribution of the corresponding vector of sample Lorenz curve ordinates, and hence those of the corresponding estimators of the Gini coefficient as well; see Beach and Davidson (1983).

This paper makes two contributions to the literature on statistical inference for GL dominance. First, we derive the asymptotic distribution of a vector of sample GL curve ordinates, interpreting it as a method-of-moments (MM) estimator, and obtain a new expression of its asymptotic variance–covariance matrix. Beach and Davidson (1983) apply the asymptotic theory of linear functions of order statistics to obtain a different expression of the same result. Since the asymptotic theory of MM estimators is standard in econometrics while that of linear functions of order statistics is not, our derivation should be more intuitive. Although the estimating function of our MM estimator is not differentiable with respect to the parameter vector, we can apply empirical process methods to derive the asymptotic distribution; see Andrews (1994).

Second, we propose a simple simulation-based test for GL dominance (and for multiple inequality restrictions in general) that is consistent and asymptotically has the correct size. Let $\theta_0 \in \Re^k$ be a parameter vector, e.g., the difference between two vectors of ordinates from two GL curves. For testing a joint hypothesis

$H : \theta_0 \geq 0$, two formulations are possible. A *multivariate one-sided testing problem* is

$$H_0 : \theta_0 = 0 \quad \text{vs.} \quad H_1 : \theta_0 \geq 0.$$

This formulation is unreasonable if it is possible that $\theta_0 \ngeq 0$. Instead, we focus on a *multivariate inequality testing problem*

$$H_0 : \theta_0 \geq 0 \quad \text{vs.} \quad H_1 : \theta_0 \ngeq 0,$$

and propose simultaneously testing for $j = 1, \ldots, k$,

$$H_{0,j} : \theta_{0,j} \geq 0 \quad \text{vs.} \quad H_{1,j} : \theta_{0,j} < 0.$$

A simultaneous test accepts H_0 if it accepts $H_{0,1}, \ldots, H_{0,k}$. Let t_n be a vector of the t statistics for testing these separate hypotheses given a sample of size n. Let $t_{n,\min}$ be the minimum component of t_n. Then a simultaneous one-sided t test accepts H_0 if $t_{n,\min}$ is above a critical value (hence we call it a $t_{\min}$ test). Unless the t statistics are asymptotically independent, it is difficult to derive the asymptotic distribution of $t_{n,\min}$ analytically. Given the asymptotic distribution of t_n, however, we can simulate the asymptotic distribution of $t_{n,\min}$ under the least favorable case in H_0, i.e., $\theta_0 = 0$, and evaluate the asymptotic p-value. Aura (2000) proposes a similar test for multivariate one-sided testing problems.

Alternatively, one can extend the classical asymptotic tests to tests for multiple inequality restrictions, in which case the test statistics have an asymptotic $\bar{\chi}^2$ distribution, a mixture of χ^2 distributions with different degrees of freedom, under the least favorable case in H_0; see Kodde and Palm (1986) and Wolak (1989). Xu (1997) and Dardanoni and Forcina (1999) propose a distance (Wald) test for GL and Lorenz dominance respectively. An interesting question here is, which of the two tests is better? (Both tests usually require simulation, because the asymptotic distributions of the test statistics are nonstandard.) We perform Monte Carlo experiments and find that the $t_{\min}$ test tends to be more powerful against crossing curves, while the $\bar{\chi}^2$ test tends to be more powerful against other alternative hypotheses. The results coincide with the analytical result by Goldberger (1992), who considers testing inequality restrictions on the mean vector of a bivariate normal distribution.

A notable feature of these testing procedures is that they are applicable even when only grouped data are available. As an example, we apply both the $t_{\min}$ and $\bar{\chi}^2$ tests to the publicly available grouped data of the National Survey of Family Income and Expenditure in Japan. The tests accept the null hypothesis that income distribution in Japan improved from 1979 to 1994, and reject the null hypothesis that it worsened from 1979 to 1994. We also find that income distribution in Japan worsened from 1994 to 1999, because the average real income decreased and income inequality increased.

The plan of the paper is as follows. Section 2 surveys the literature on statistical inference for Lorenz and GL dominance. Section 3 reviews the notion of GL dominance. Section 4 gives a new derivation of the asymptotic distribution of a vector of sample GL curve ordinates. Section 5 explains our simulation-based asymptotic $t_{\min}$ test. Section 6 compares alternative tests by Monte Carlo experiments. Section 7 applies the two tests to the Japanese household income data. Section 8 discusses remaining issues. Appendix A contains proofs. Appendix B summarizes the data used in application.

2 LITERATURE

Gail and Gastwirth (1978) apply the asymptotic theory of linear functions of order statistics to derive the asymptotic distribution of a vector of sample GL curve ordinates; see Moore (1968) for an elementary proof of asymptotic normality of linear functions of order statistics. Beach and Davidson (1983) show that the asymptotic distribution is "distribution-free" in that it depends only on the group means and variances of the population. Zheng (1999) derives the same result using Bahadur's representation of sample quantiles. Zheng (2002) extends the result to non-simple random samples.

Let $\theta_0 \in \Re^k$ be the difference between two vectors of ordinates from two Lorenz or GL curves. Beach and Davidson (1983) propose a Wald test for testing

$$H_0 : \theta_0 = 0 \quad \text{vs.} \quad H_1 : \theta_0 \neq 0. \tag{1}$$

This is a test for equality of two curves, not for dominance.

For this two-sided problem, Bishop, Formby, and Thistle (1989) propose simultaneously testing for $j = 1, \ldots, k$,

$$H_{0,j} : \theta_{0,j} = 0 \quad \text{vs.} \quad H_{1,j} : \theta_{0,j} \neq 0. \tag{2}$$

The test accepts $H_0 : \theta_0 = 0$ if it accepts $H_{0,1}, \ldots, H_{0,k}$. Let $\hat{\theta}_n$ be an estimator of θ_0 such that $\sqrt{n}\left(\hat{\theta}_n - \theta_0\right) \to_d$ $\mathrm{N}(0, \Sigma)$. Let $m_\alpha(k, d)$ be the $(1-\alpha)$-quantile of the studentized maximum modulus (SMM) distribution with parameter k and d degrees of freedom. If Σ is diagonal, i.e., $\hat{\theta}_{n,1}, \ldots, \hat{\theta}_{n,k}$ are asymptotically independent, then under H_0,

$$\lim_{n\to\infty} \Pr\left[\max_{j\in\{1,\ldots,k\}} \frac{\left|\sqrt{n}\hat{\theta}_{n,j}\right|}{\sigma_j} > m_\alpha(k,\infty)\right] = \alpha.$$

Even if Σ is not diagonal, under H_0, we have

$$\lim_{n\to\infty} \Pr\left[\max_{j\in\{1,\ldots,k\}} \frac{\left|\sqrt{n}\hat{\theta}_{n,j}\right|}{\sigma_j} > m_\alpha(k,\infty)\right] \leq \alpha,$$

as in Stoline and Ury (1979, p. 89). We replace Σ with a consistent estimator in practice. Since Σ is usually not diagonal, the test tends to be conservative, i.e., the actual size is less than the nominal size. Moreover, the test may be inefficient because it ignores the covariances.

Gastwirth and Gail (1985) propose multivariate one-sided tests for Lorenz dominance, i.e., they consider testing

$$H_0 : \theta_0 = 0 \quad \text{vs.} \quad H_1 : \theta_0 \geq 0. \tag{3}$$

One of their test statistics (their T_2) is the difference between the sums of ordinates of the two sample Lorenz curves. Bishop, Chakraborti, and Thistle (1989) extend it to a test for GL dominance. The problem of multivariate one-sided tests in this context, however, is that neither H_0 nor H_1 covers crossing curves, which we cannot assume away. If the asymptotic power of a test against some crossing curves is 1, then the test mistakenly accepts H_1 with probability 1 as the sample size goes to infinity.

Aura (2000) essentially considers multivariate one-sided testing problems, and proposes simultaneously testing for $j = 1, \ldots, k$,

$$H_{0,j} : \theta_{0,j} = 0 \quad \text{vs.} \quad H_{1,j} : \theta_{0,j} > 0. \tag{4}$$

The test accepts $H_0 : \theta_0 = 0$ if it accepts $H_{0,1}, \ldots, H_{0,k}$. Let t_n be a vector of t statistics for testing these separate hypotheses given a sample of size n. Let $t_{n,\max}$ be the maximum component of t_n. His test essentially accepts H_1 if $t_{n,\max}$ exceeds a simulated critical value. It may happen that $t_{n,\min}$, the minimum component of t_n, is also small, which suggests that some components of θ_0 are negative. Hence he actually proposes a bivariate test statistic (t_+, t_-), where $t_+ := \max\{0, t_{n,\max}\}$ and $t_- := \min\{0, t_{n,\min}\}$.

Xu (1997) considers testing the null of GL dominance against the alternative of no dominance, i.e.,

$$H_0 : \theta_0 \geq 0 \quad \text{vs.} \quad H_1 : \theta_0 \ngeq 0. \tag{5}$$

He applies a distance (Wald) test for multiple inequality restrictions discussed in Kodde and Palm (1986) and Wolak (1989). Let $\hat{\theta}_n$ be an unrestricted estimator of θ_0 such that $\sqrt{n}\left(\hat{\theta}_n - \theta_0\right) \to_d \mathrm{N}(0, \Sigma)$. Let $\hat{\Sigma}_n$ be a consistent estimator of Σ. Let $\tilde{\theta}_n$ be a minimum distance estimator of θ_0 such that

$$\begin{aligned}\tilde{\theta}_n := \arg\min_{\theta} \quad & \left(\hat{\theta}_n - \theta\right)' \hat{\Sigma}_n^{-1} \left(\hat{\theta}_n - \theta\right) \\ \text{s.t.} \quad & \theta \geq 0.\end{aligned}$$

The distance test statistic is

$$D_n := n\left(\hat{\theta}_n - \tilde{\theta}_n\right)' \hat{\Sigma}_n^{-1} \left(\hat{\theta}_n - \tilde{\theta}_n\right).$$

Kodde and Palm (1986) show that under the least favorable case in H_0, i.e., $\theta_0 = 0$, D_n has an asymptotic $\bar{\chi}^2$ distribution such that for all $c \in \Re$,

$$\lim_{n\to\infty} \Pr[D_n \geq c] = \sum_{j=0}^{k} p_j(\Sigma) \Pr\left[\chi^2(j) \geq c\right],$$

where $p_j(\Sigma)$ is the probability that j components of $\tilde{\theta}_n$ are 0 given Σ. The result is inconvenient because the asymptotic distribution depends on Σ; hence Kodde and Palm (1986) give the upper and lower bounds for the critical values. Dardanoni and Forcina (1999) extend the distance test to comparison of m Lorenz curves, where $m \geq 2$. They consider three hypotheses:

- $H_0 : L_1 = \cdots = L_m$,
- $H_1 : L_1 \geq \cdots \geq L_m$,
- H_2 : no restrictions,

and propose distance tests for testing (i) H_0 vs. H_2, (ii) H_0 vs. H_1, (iii) H_1 vs. H_2, and (iv) $H_2 - H_1$ vs. H_2.

For continuous distributions, GL dominance is equivalent to the second-order stochastic dominance (SSD); see Foster and Shorrocks (1988) and Yitzhaki and Olkin (1991). Several tests for the SSD exist in the literature; see Davidson and Duclos (2000) and references there. Since those tests do not consider GL curves directly, they do not lead to statistical inference for Lorenz curves and Gini coefficients. Moreover, those tests usually require micro data.

Testing for GL or stochastic dominance is a special case of testing multiple inequality hypotheses, on which Maasoumi (2001) gives an excellent survey.

3 GENERALIZED LORENZ DOMINANCE

3.1 Generalized Lorenz Curves

Let X be a positive random variable. Let $F : \Re \to [0,1]$ be the cumulative distribution function (cdf) of X. Let for all $\alpha \in [0,1]$, x_α be the α-quantile of X defined as $x_\alpha := \inf\{x \in \Re_+ : F(x) \geq \alpha\}$. Let $\mu := \mathrm{E}(X)$.

Definition 1 *The* Lorenz curve *of X is $L : [0,1] \to [0,1]$ such that for all $\alpha \in [0,1]$,*

$$L(\alpha) := \frac{\mathrm{E}([X \leq x_\alpha]X)}{\mu}. \tag{6}$$

Definition 2 *The* generalized Lorenz (GL) curve *of X is $GL : [0,1] \to [0,\mu]$ such that for all $\alpha \in [0,1]$,*

$$GL(\alpha) := \mathrm{E}([X \leq x_\alpha]X). \tag{7}$$

Let $F_1(.)$ and $F_2(.)$ be cdfs. We say that $F_1(.)$ *GL dominates* $F_2(.)$ if the GL curve of $F_1(.)$ lies above that of $F_2(.)$.

3.2 Income Distributions and Social Welfare

3.2.1 Social welfare functions

Let $y \in \Re^n$ be a distribution of income (or consumption, wealth, etc.) among n households in an economy. Let $W : \Re^n \to \Re$ be a social welfare function (SWF) that depends solely on y.

Definition 3 $B \in \Re_+^{n \times n}$ *is* bistochastic *if the components in each row and column add up to 1 respectively.*

Definition 4 $W(.)$ *is* Schur-concave (S-concave) *if for all y and for all bistochastic matrices B,*

$$W(By) \geq W(y).$$

S-concave functions are symmetric, i.e., for all y and for all permutation matrices P, $W(Py) = W(y)$; see Berge (1963, p. 220). S-concave SWFs satisfy the Pigou–Dalton (P–D) principle of transfers. To be precise, an SWF is strictly S-concave if and only if it satisfies the P–D principle; see Sen (1997, p. 134). For example, symmetric quasiconcave functions are S-concave; see Dasgupta, Sen, and Starrett (1973, p. 183).

3.2.2 Generalized Lorenz dominance and social welfare

Assume that $y \geq 0$ and that it is ordered. The GL curve of y is for all $\alpha \in [0,1]$,

$$GL(\alpha) := \frac{1}{n} \sum_{i=1}^{[\alpha n]} y_i,$$

where [.] rounds up a real number to an integer. We say that y GL dominates y' if the GL curve of y lies above that of y'.

Theorem 1 (Shorrocks (1983)) *$W(y) \geq W(y')$ for all increasing and S-concave $W(.)$ if and only if y GL dominates y'.*

Suppose that $W(.)$ is invariant to replication of the population. Then the theorem holds even when the dimensions of y and y' differ.

4 SAMPLE GENERALIZED LORENZ CURVE ORDINATES

4.1 Sample Generalized Lorenz Curves

Let $(X_1, \ldots, X_n)$ be a sample of size n. Let $\hat{F}_n : \Re \to [0,1]$ be the empirical cdf given the sample, i.e., for all $x \in \Re$,

$$\hat{F}_n(x) := \frac{1}{n} \sum_{i=1}^{n} [X_i \leq x].$$

Let $X_{(1)}, \ldots, X_{(n)}$ be the order statistics. Let for all $\alpha \in [0,1]$, $\hat{x}_{n,\alpha}$ be the sample α-quantile, i.e.,

$$\begin{aligned} \hat{x}_{n,\alpha} &:= \inf \left\{ x \in \Re_+ : \hat{F}_n(x) \geq \alpha \right\} \\ &= \inf \left\{ x \in \Re_+ : \frac{1}{n} \sum_{i=1}^{n} [X_i \leq x] \geq \alpha \right\} \\ &= X_{([\alpha n])}. \end{aligned}$$

Let $\hat{\mu}_n$ be the sample mean.

Definition 5 *The* sample GL curve *given $(X_1, \ldots, X_n)$ is $\hat{GL}_n : [0,1] \to [0, \hat{\mu}_n]$ such that for all $\alpha \in [0,1]$,*

$$\hat{GL}_n(\alpha) := \frac{1}{n} \sum_{i=1}^{n} [X_i \leq \hat{x}_{n,\alpha}] X_i. \tag{8}$$

4.2 Consistency

Gail and Gastwirth (1978) prove pointwise consistency of the sample GL curve in their proof of pointwise consistency of the sample Lorenz curve.

Theorem 2 *Suppose that*

1. $X_1, \ldots, X_n$ *are independent and identically distributed (iid),*

2. $\mathrm{E}(|X_1|) < \infty$,

3. $F(.)$ *is strictly increasing and* C^0 *at* x_α.

Then

$$\lim_{n \to \infty} \hat{G}L_n(\alpha) = GL(\alpha) \quad a.s.$$

Proof. See Gail and Gastwirth (1978, p. 788). □

The first condition holds for simple random sampling (SRS) and probability-proportional-to-size (PPS) sampling *with* replacement. Given the third condition, which implies an infinite population, it also holds for SRS and PPS sampling *without* replacement, including systematic sampling with randomized order of the population. It does not hold for stratified sampling, however.

Suppose that there are m strata. Let for $h = 1, \ldots, m$, w_h be the relative size of the hth stratum and $F_h(.)$ be the cdf of X in the hth stratum. Then for all $x \in \Re$,

$$F(x) = \sum_{h=1}^{m} w_h F_h(x).$$

Let for $h = 1, \ldots, m$, $\left(X_1^h, \ldots, X_{n_h}^h\right)$ be the subsample from the hth stratum of size n_h and $\hat{F}_{h,n_h}(.)$ be the empirical cdf given the subsample. A consistent estimator of $F(.)$ is $\hat{F}_n(.)$ such that for all $x \in \Re$,

$$\begin{aligned} \hat{F}_n(x) &:= \sum_{h=1}^{m} w_h \hat{F}_{h,n_h}(x) \\ &= \sum_{h=1}^{m} \sum_{i=1}^{n_h} \frac{w_h}{n_h} \left[X_i^h \leq x\right]. \end{aligned}$$

This equals the empirical cdf of the whole sample if for $h = 1, \ldots, m$, $n_h = w_h n$ (proportional allocation).

By definition, for all $\alpha \in [0,1]$, the sample α-quantile is

$$\begin{aligned} \hat{x}_{n,\alpha} &:= \inf\left\{x \in \Re_+ : \hat{F}_n(x) \geq \alpha\right\} \\ &= \inf\left\{x \in \Re_+ : \sum_{h=1}^{m}\sum_{i=1}^{n_h} \frac{w_h}{n_h}\left[X_i^h \leq x\right] \geq \alpha\right\}. \end{aligned}$$

Let for $h = 1, \ldots, m$, $GL_h(.)$ be the GL curve of X in the hth stratum. Then for all $\alpha \in [0,1]$,

$$GL(\alpha) = \sum_{h=1}^{m} w_h GL_h(\alpha).$$

Let for $h = 1, \ldots, m$, $\hat{GL}_{h,n_h}(.)$ be the sample GL curve given the subsample from the hth stratum. A consistent estimator of $GL(.)$ is $\hat{GL}_n(.)$ such that for all $\alpha \in [0,1]$,

$$\begin{aligned} \hat{GL}_n(\alpha) &:= \sum_{h=1}^{m} w_h \hat{GL}_{h,n_h}(\alpha) \\ &= \sum_{h=1}^{m}\sum_{i=1}^{n_h} \frac{w_h}{n_h}\left[X_i^h \leq \hat{x}_{n,\alpha}\right] X_i^h. \end{aligned}$$

4.3 Asymptotic Distribution

Let $0 < \alpha_1 < \cdots < \alpha_k = 1$. Let for $j = 1, \ldots, k$, x_j be the α_j-quantile of X. The corresponding GL curve ordinates of X are for $j = 1, \ldots, k$,

$$GL_j := \mathrm{E}([X \leq x_j]X). \tag{9}$$

Let for $j = 1, \ldots, k$, $\hat{x}_{n,j}$ be the sample α_j-quantile. The corresponding sample GL curve ordinates given $(X_1, \ldots, X_n)$ are for $j = 1, \ldots, k-1$,

$$\hat{GL}_{n,j} := \frac{1}{n}\sum_{i=1}^{n}[X_i \leq \hat{x}_{n,j}]X_i, \tag{10}$$

and

$$\hat{GL}_{n,k} := \frac{1}{n}\sum_{i=1}^{n} X_i. \tag{11}$$

Beach and Davidson (1983) derive the asymptotic joint distribution of sample GL curve ordinates using the asymptotic theory of linear functions of order statistics, noting that for $j = 1, \ldots, k$,

$$\hat{GL}_{n,j} = \frac{1}{n}\sum_{i=1}^{[\alpha_j n]} X_{(i)}.$$

Since such asymptotic theory is not standard in econometrics, we give an alternative derivation, noting that a vector of sample GL curve ordinates is a method-of-moments (MM) estimator.

Let

$$\theta_0 := \begin{pmatrix} x_1 \\ \vdots \\ x_{k-1} \\ GL_1 \\ \vdots \\ GL_k \end{pmatrix}, \quad \hat{\theta}_n := \begin{pmatrix} \hat{x}_{n,1} \\ \vdots \\ \hat{x}_{n,k-1} \\ \hat{GL}_{n,1} \\ \vdots \\ \hat{GL}_{n,k} \end{pmatrix}.$$

Let $\Theta \subset \Re_+^{2k-1}$ be the parameter space. Given $\theta \in \Theta$, let for $i = 1, \ldots, n$,

$$m(X_i; \theta) := \begin{pmatrix} [X_i \leq x_1] - \alpha_1 \\ \vdots \\ [X_i \leq x_{k-1}] - \alpha_{k-1} \\ [X_i \leq x_1] X_i - GL_1 \\ \vdots \\ [X_i \leq x_{k-1}] X_i - GL_{k-1} \\ X_i - GL_k \end{pmatrix}. \tag{12}$$

Assume that $X_1, \ldots, X_n$ are iid. Let $m_0 : \Theta \to \Re^{2k-1}$ be such that for all $\theta \in \Theta$,

$$m_0(\theta) := \mathrm{E}(m(X_1; \theta)). \tag{13}$$

Then we have a moment restriction such that

$$m_0(\theta_0) = 0. \tag{14}$$

Let $\bar{m}_n(.)$ be the sample analog of $m_0(.)$, i.e., for all $\theta \in \Theta$,

$$\bar{m}_n(\theta) := \frac{1}{n} \sum_{i=1}^{n} m(X_i; \theta). \tag{15}$$

Note that for $j = 1, \ldots, k-1$,

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^{n} [X_i \leq \hat{x}_{n,j}] &= \frac{[\alpha_j n]}{n} \\ &= \alpha_j + \frac{[\alpha_j n] - \alpha_j n}{n}. \end{aligned}$$

Hence

$$\bar{m}_n\left(\hat{\theta}_n\right) = O\left(n^{-1}\right), \tag{16}$$

i.e., $\hat{\theta}_n$ is an MM estimator of θ_0. Theorem 2 essentially gives a sufficient condition for $\hat{\theta}_n$ to be consistent for θ_0.

Although $m(.;.)$ is not differentiable with respect to θ, we can apply empirical process methods to derive the asymptotic distribution of $\hat{\theta}_n$; see Andrews (1994). Let $\nu_n(.)$ be a $(2k-1)$-variate empirical process on Θ given $(X_1, \ldots, X_n)$ such that for all $\theta \in \Theta$,

$$\nu_n(\theta) := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (m(X_i;\theta) - \mathrm{E}(m(X_i;\theta))).$$

Definition 6 $\{\nu_n(.)\}_{n=1}^{\infty}$ *is* stochastically equicontinuous (uniformly on Θ) *if for all* $\epsilon > 0$,

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \Pr{}^* \left[\sup_{\theta, \theta' \in \Theta, d(\theta,\theta') < \delta} \|\nu_n(\theta) - \nu_n(\theta')\| > \epsilon \right] = 0,$$

where $\Pr^*[.]$ *is the outer probability,* $d(.,.)$ *is a metric on* Θ*, and* $\|.\|$ *is a norm on* $\Re^{2k-1}$.

Note that a sequence of multivariate empirical processes is stochastically equicontinuous if the notion applies to each component; see Andrews (1994, p. 2267).

Theorem 3 *Suppose that*

1. $X_1, \ldots, X_n$ *are iid,*
2. $\mathrm{E}\left(|X_1|^2\right) < \infty$,
3. $F(.)$ *is strictly increasing and* C^1 *on its support,*
4. $\{\nu_n(.)\}_{n=1}^{\infty}$ *is stochastically equicontinuous.*

Then

$$\sqrt{n}\left(\hat{\theta}_n - \theta\right) \to_d \mathrm{N}\left(0, J^{-1} V J^{-1\prime}\right),$$

where

$$\begin{aligned} J &:= m_0'(\theta_0), \\ V &:= \mathrm{var}(m(X_1;\theta_0)). \end{aligned}$$

Proof. See Appendix A. □

In our case, it turns out that the first two conditions are sufficient for stochastic equicontinuity of each component of $\{\nu_n(.)\}_{n=1}^{\infty}$; hence the last condition is unnecessary.

Theorem 4 *Suppose that*

1. $X_1, \ldots, X_n$ *are iid,*

2. $\mathrm{E}\left(|X_1|^2\right) < \infty.$

Then $\{\nu_n(.)\}_{n=1}^{\infty}$ *is stochastically equicontinuous.*

Proof. See Appendix A. $\square$

Since the sample GL curve ordinates are the last k components of $\hat{\theta}_n$, it is now straightforward to obtain their asymptotic joint distribution. Let

$$GL := \begin{pmatrix} GL_1 \\ \vdots \\ GL_k \end{pmatrix}, \quad \hat{G}L_n := \begin{pmatrix} \hat{G}L_{n,1} \\ \vdots \\ \hat{G}L_{n,k} \end{pmatrix}.$$

Theorem 5 *Suppose that*

1. $X_1, \ldots, X_n$ *are iid,*

2. $\mathrm{E}\left(|X_1|^2\right) < \infty,$

3. $F(.)$ *is strictly increasing and* C^1 *on its support.*

Then

$$\sqrt{n}\left(\hat{G}L_n - GL\right) \to_d \mathrm{N}(0, \Sigma),$$

where for $i, j = 1, \ldots, k$ *such that* $i \leq j$,

$$\begin{aligned} \sigma_{i,j} \quad := \quad & x_i \alpha_i (1 - \alpha_j) x_j - x_i (GL_i - \alpha_i GL_j) - (GL_i - GL_i \alpha_j) x_j \\ & + \mathrm{E}\left([X_1 \leq x_i] X_1^2\right) - GL_i GL_j. \end{aligned} \tag{17}$$

Proof. See Appendix A. $\square$

Our result clarifies the effect of two-step estimation involved in the sample GL curve. The last two terms equal $\mathrm{cov}([X_1 \leq x_i] X_1, [X_1 \leq x_j] X_1)$, which would have resulted if we knew the true quantiles. The first three terms capture the effect of using the sample quantiles instead of the true ones.

Compare our result with the corresponding result in Beach and Davidson (1983, Theorem 1). In our notation, they obtain for $i, j = 1, \ldots, k$ such that $i \leq j$,

$$\sigma_{i,j} := \alpha_i [\mathrm{var}(X | X \leq x_i) + (1 - \alpha_j)(x_i - \mu_i)(x_j - \mu_j) + (x_i - \mu_i)(\mu_j - \mu_i)], \tag{18}$$

where $\mu_i := \mathrm{E}(X|X \leq x_i)$. It is tedious but straightforward to show that the two are equivalent.

Again, Theorem 5 does not hold for stratified samples. Suppose that there are m strata. Let for $h = 1, \ldots, m$, w_h be the relative size of the hth stratum and GL_h be a vector of GL curve ordinates of the hth stratum. Then

$$GL = \sum_{h=1}^{m} w_h GL_h.$$

Let for $h = 1, \ldots, m$, GL_{h,n_h} be a vector of sample GL curve ordinates given the subsample from the hth stratum of size n_h. A consistent estimator of GL is

$$\hat{GL}_n := \sum_{h=1}^{m} w_h \hat{GL}_{h,n_h}.$$

Suppose that Theorem 5 applies to each subsample, i.e., for $h = 1, \ldots, m$,

$$\sqrt{n_h}\left(\hat{GL}_{h,n_h} - GL_h\right) \to_d \mathrm{N}(0, \Sigma_h).$$

Assume that $\lim_{n\to\infty} n_h/n = t_h$. Then for $h = 1, \ldots, m$,

$$\sqrt{n}\left(\hat{GL}_{h,n_h} - GL_h\right) \to_d \mathrm{N}\left(0, \frac{\Sigma_h}{t_h}\right).$$

Assume that $\hat{GL}_{1,n_1}, \ldots, \hat{GL}_{m,n_m}$ are independent. Then

$$\sqrt{n}\left(\hat{GL}_n - GL\right) \to_d \mathrm{N}\left(0, \sum_{h=1}^{m} \frac{w_h^2 \Sigma_h}{t_h}\right).$$

The result is the same as that in Zheng (2002, p. 1238), who also derives results for cluster samples and multistage samples.

4.4 Covariance Matrix Estimation

We can consistently estimate Σ by replacing the parameters associated with $F(.)$ with those associated with $\hat{F}_n(.)$. Let $\hat{\Sigma}_n$ be such an estimator. Then for $i, j = 1, \ldots, k$ such that $i \leq j$,

$$\begin{aligned}
\hat{\sigma}_{n,i,j} \quad := \quad & \hat{x}_{n,i}\hat{\alpha}_i(1-\hat{\alpha}_j)\hat{x}_{n,j} \\
& -\hat{x}_{n,i}\left(\hat{GL}_{n,i} - \hat{\alpha}_i \hat{GL}_{n,j}\right) - \left(\hat{GL}_{n,i} - \hat{GL}_{n,i}\hat{\alpha}_j\right)\hat{x}_{n,j} \\
& +\hat{\mathrm{E}}_n\left([X_1 \leq x_i]X_1^2\right) - \hat{GL}_{n,i}\hat{GL}_{n,j}. \qquad (19)
\end{aligned}$$

5 TESTING FOR GENERALIZED LORENZ DOMINANCE

5.1 Multivariate One-Sided Tests and Multivariate Inequality Tests

Let $F_1(.)$ and $F_2(.)$ be cdfs. Let GL_1 and GL_2 be vectors of GL curve ordinates associated with $F_1(.)$ and $F_2(.)$ respectively. Let $\theta_0 := GL_1 - GL_2$. Then $\theta_0 \geq 0$ if $F_1(.)$ GL dominates $F_2(.)$. Goldberger (1992) distinguishes the following two formulations for testing multiple inequality hypotheses.

Definition 7 *A* multivariate one-sided testing problem *is*

$$H_0 : \theta_0 = 0 \quad vs. \quad H_1 : \theta_0 \geq 0.$$

Definition 8 *A* multivariate inequality testing problem *is*

$$H_0 : \theta_0 \geq 0 \quad vs. \quad H_1 : \theta_0 \ngeq 0.$$

In general, the first formulation is better for asserting $\theta_0 \geq 0$. A drawback of this formulation, however, is that neither H_0 nor H_1 covers crossing GL curves. This is a serious drawback in our context, because it is quite possible that two GL curves cross and hence the two distributions are incomparable. Thus we choose the second formulation. Note that now we assert GL dominance by accepting H_0. Such a conclusion is weak, because the power of the test is not under our direct control.

5.2 $\theta_{\min}$ Test

Let $\hat{G}L_{1,n_1}$ and $\hat{G}L_{2,n_2}$ be vectors of sample GL curve ordinates of two independent random samples of sizes n_1 and n_2 from $F_1(.)$ and $F_2(.)$ respectively. By Theorem 5, for $i = 1, 2$,

$$\sqrt{n_i}\left(\hat{G}L_{i,n_i} - GL_i\right) \to_d \mathrm{N}(0, \Sigma_i).$$

Let $n := n_1 + n_2$. Assume that for $i = 1, 2$, $\lim_{n\to\infty} n_i/n = t_i$. Then for $i = 1, 2$,

$$\sqrt{n}\left(\hat{G}L_{i,n_i} - GL_i\right) \to_d \mathrm{N}\left(0, \frac{\Sigma_i}{t_i}\right).$$

Let $\hat{\theta}_n := \hat{G}L_{1,n_1} - \hat{G}L_{2,n_2}$. Since $\hat{G}L_{1,n_1}$ and $\hat{G}L_{2,n_2}$ are independent,

$$\sqrt{n}\left(\hat{\theta}_n - \theta_0\right) \to_d \mathrm{N}\left(0, \frac{\Sigma_1}{t_1} + \frac{\Sigma_2}{t_2}\right),$$

or

$$\hat{\theta}_n \sim_a \mathrm{N}\left(\theta_0, \frac{\Sigma_1}{n_1} + \frac{\Sigma_2}{n_2}\right). \tag{20}$$

Thus, given Σ_1 and Σ_2, we know the asymptotic distribution of $\hat{\theta}_n$ under the least favorable case in H_0, i.e., $\theta_0 = 0$.

Let $\theta_{\min}$ be the minimum component of θ_0. Then we can write the multivariate inequality testing problem as

$$H_0 : \theta_{\min} \geq 0 \quad \text{vs.} \quad H_1 : \theta_{\min} < 0.$$

Hence a natural test statistic is $\hat{\theta}_{n,\min}$, the minimum component of $\hat{\theta}_n$.

Let $F(.)$ be the cdf of the minimum component of $X \sim \mathrm{N}(0, \Sigma_1/n_1 + \Sigma_2/n_2)$. Then under the least favorable case in H_0, i.e., $\theta_0 = 0$,

$$\hat{\theta}_{n,\min} \sim_a F(.). \tag{21}$$

It is difficult to derive $F(.)$ analytically. Given Σ_1 and Σ_2, however, we can draw from $\mathrm{N}(0, \Sigma_1/n_1 + \Sigma_2/n_2)$ and simulate $F(.)$; thus we can evaluate the asymptotic p-value. In practice, we replace Σ_1 and Σ_2 with their consistent estimators. We call this a $\theta_{\min}$ test.

5.3 $t_{\min}$ Test

Alternatively, we can simultaneously test for $j = 1, \ldots, k$,

$$H_{0,j} : \theta_{0,j} \geq 0 \quad \text{vs.} \quad H_{1,j} : \theta_{0,j} < 0.$$

This testing problem can arise from a multivariate one-sided testing problem

$$H_0 : \theta_0 = 0 \quad \text{vs.} \quad H_1 : \theta_0 \leq 0.$$

where we accept $H_0 : \theta_0 = 0$ if we accept $H_{0,1}, \ldots, H_{0,k}$, as well as from a multivariate inequality testing problem

$$H_0 : \theta_0 \geq 0 \quad \text{vs.} \quad H_1 : \theta_0 \ngeq 0,$$

where we accept $H_0 : \theta_0 \geq 0$ if we accept $H_{0,1}, \ldots, H_{0,k}$.

A natural test for this simultaneous testing problem is a simulation-based simultaneous asymptotic one-sided t test. Let

$$t_n := \operatorname{diag}\left(\hat{\Sigma}_n\right)^{-1/2} \sqrt{n}\hat{\theta}_n. \tag{22}$$

Let $t_{n,\min}$ be the minimum component of t_n. Then a simultaneous one-sided t test accepts H_0 if $t_{n,\min}$ is above a critical value. Again, it is difficult to derive the asymptotic distribution of $t_{n,\min}$ analytically; hence we evaluate the asymptotic p-value by simulation. We call this a $t_{\min}$ test.

Aura (2000) essentially proposes the same procedure for multivariate one-sided testing problems. Note that if $t_{n,\min} < c$, where c is a simulated critical value, then we accept $H_1 : \theta_0 \leq 0$ for multivariate one-sided testing problems, while we accept only $H_1 : \theta_0 \ngeq 0$ for multivariate inequality testing problems.

5.4 Asymptotic Properties

The two proposed tests both asymptotically have the correct sizes and are consistent. We state the results as theorems.

Theorem 6 *For any significance level α, the asymptotic sizes of the $\theta_{\min}$ and $t_{\min}$ tests are α.*

Proof. See Appendix A. □

Theorem 7 *Both $\theta_{\min}$ and $t_{\min}$ tests are consistent.*

Proof. See Appendix A. □

6 MONTE CARLO EXPERIMENTS

6.1 Design of Experiments

We perform Monte Carlo experiments to compare our tests with the $\bar{\chi}^2$ test. To see the power of the tests against crossing curves, we test Lorenz (not GL) dominance of one Singh–Maddala (S–M) distribution over another. Let $F_1(.)$ and $F_2(.)$ be the cdfs of S–M distributions, i.e., for $i = 1, 2$, for all $x \geq 0$,

$$F_i(x) := 1 - \frac{1}{[1 + (x/b_i)^{a_i}]^{q_i}}.$$

Wilfling and Krämer (1993) show that[1]

[1]We say that $F_1(.)$ Lorenz dominates $F_2(.)$ if the Lorenz curve of $F_1(.)$ lies above that of $F_2(.)$. Wilfling and Krämer (1993, p. 53) define it in the opposite way, which is unconventional.

- $F_1(.)$ Lorenz dominates $F_2(.)$ if and only if $a_1 \geq a_2$ and $a_1 q_1 \geq a_2 q_2$,
- $F_2(.)$ Lorenz dominates $F_1(.)$ if and only if $a_1 \leq a_2$ and $a_1 q_1 \leq a_2 q_2$.

Hence two Lorenz curves cross otherwise.

Let L_1 and L_2 be vectors of Lorenz curve ordinates of $F_1(.)$ and $F_2(.)$ respectively. Consider testing

$$H_0 : L_1 \geq L_2 \quad \text{vs.} \quad H_1 : L_1 \ngeq L_2.$$

Following Dardanoni and Forcina (1999), we fix $(a_2, b_2, q_2) := (1.697, 1, 8.368)$ and consider the following seven cases:

- Case 1: $(a_1, b_1, q_1) := (1.697, 1, 8.368)$,
- Case 2: $(a_1, b_1, q_1) := (1.697 + .07, 1, 8.368)$,
- Case 3: $(a_1, b_1, q_1) := (1.697 + .14, 1, 8.368)$,
- Case 4: $(a_1, b_1, q_1) := (1.697 - .07, 1, 8.368)$,
- Case 5: $(a_1, b_1, q_1) := (1.697 - .14, 1, 8.368)$,
- Case 6: $(a_1, b_1, q_1) := (1.817, 1, 4.1996)$,
- Case 7: $(a_1, b_1, q_1) := (2.057, 1, 2.1397)$.

In addition, we switch L_1 and L_2 in Case 6 and 7:

- Case 8: $(a_1, b_1, q_1) := (1.697, 1, 8.368)$, $(a_2, b_2, q_2) := (1.817, 1, 4.1996)$,
- Case 9: $(a_1, b_1, q_1) := (1.697, 1, 8.368)$, $(a_1, b_1, q_1) := (2.057, 1, 2.1397)$.

We have $L_1 = L_2$ in Case 1, $L_1 \geq L_2$ in Case 2 and 3, $L_1 \leq L_2$ in Case 4 and 5, and crossing Lorenz curves in Case 6–9.

Given the asymptotic distribution of a vector of sample GL curve ordinates, we can apply the delta method to obtain the asymptotic distribution of the corresponding vector of sample Lorenz curve ordinates.

Let $GL \in \Re^k$ be a vector of GL curve ordinates and L be the corresponding vector of Lorenz curve ordinates, i.e.,

$$L = \begin{pmatrix} \frac{GL_1}{GL_k} \\ \vdots \\ \frac{GL_{k-1}}{GL_k} \end{pmatrix}.$$

Let $\hat{GL}_n$ be a vector of sample GL curve ordinates such that $\sqrt{n}\left(\hat{GL}_n - GL\right) \to_d \mathrm{N}(0, \Sigma)$ and $\hat{L}_n$ be the corresponding vector of sample Lorenz curve ordinates. By the delta method,

$$\sqrt{n}\left(\hat{L}_n - L\right) \to_d \mathrm{N}(0, J\Sigma J'),$$

where

$$J := \begin{bmatrix} \frac{1}{GL_k} & & 0 & -\frac{GL_1}{GL_k^2} \\ & \ddots & & \vdots \\ 0 & & \frac{1}{GL_k} & -\frac{GL_{k-1}}{GL_k^2} \end{bmatrix}.$$

In the experiments, we set $k = 10$ and try three sample sizes: 1,000, 2,000, and 4,000. The simulated size and power of the tests are the relative frequencies of asymptotic p-values less than the significance level (fixed at .05) in 10,000 Monte Carlo replications. To evaluate the asymptotic p-value in each replication,[2] we simulate the asymptotic distribution of the test statistic under the least favorable case in H_0 based on 10,000 draws from the $(k-1)$-variate normal distribution with mean 0 and variance-covariance matrix $\hat{J}_n \hat{\Sigma}_n \hat{J}_n'$ (the sample analog of $J\Sigma J'$) or the associated correlation coefficient matrix. For the $\theta_{\min}$ and $t_{\min}$ tests, we simulate the distribution of the minimum component from these draws. For the $\bar{\chi}^2$ test, we simulate the weights to mix χ^2 distributions with different degrees of freedom from these draws. We use Ox 3.20 by Doornik (2001) for computation.

6.2 Results

Table 1 summarizes the results of the experiments. We confirm that

- the three tests asymptotically have the correct size (Case 1),
- Case 1 is the least favorable case under H_0 (Case 2 and 3),
- the three tests are consistent (Case 4–9).

More interestingly, we find that

[2]For the $\bar{\chi}^2$ test, we do not have to evaluate the asymptotic p-value if the test statistic exceeds the bounds given by Kodde and Palm (1986).

Table 1: Simulated Size and Power of the Tests (Case 1–9)

Test	n	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
$\theta_{\min}$	1,000	.050	.004	.000	.285	.716	.113	.527	.037	.039
	2,000	.050	.001	.000	.465	.932	.173	.824	.040	.069
	4,000	.047	.000	.000	.701	.998	.290	.983	.044	.172
$t_{\min}$	1,000	.050	.004	.000	.286	.733	.083	.408	.111	.474
	2,000	.051	.001	.000	.465	.939	.131	.747	.177	.767
	4,000	.042	.000	.000	.707	.999	.237	.969	.293	.969
$\bar{\chi}^2$	1,000	.050	.005	.000	.291	.739	.079	.396	.111	.457
	2,000	.050	.001	.000	.478	.948	.119	.727	.160	.752
	4,000	.049	.000	.000	.734	.999	.222	.966	.273	.962

Note: The numbers are rejection rates in 10,000 Monte Carlo replications.

- the power of the $\theta_{\min}$ test changes drastically if we switch the crossing curves (Case 6–9),
- Compared with the $\bar{\chi}^2$ test, the $t_{\min}$ test is less powerful against non-crossing curves (Case 4 and 5), but more powerful against crossing curves (Case 6–9).

Thus we conclude that the $t_{\min}$ test is preferable to the $\theta_{\min}$ test, and that the $t_{\min}$ and $\bar{\chi}^2$ tests are complementary. The latter coincides with the analytical result by Goldberger (1992), who considers testing inequality restrictions on the mean vector of a bivariate normal distribution.

7 APPLICATION

7.1 Income Distributions in Japan

According to the National Survey of Family Income and Expenditure, (before-tax) income inequality in Japan measured by the sample Lorenz curve worsened, while real income distribution measured by the sample GL curve improved from 1979 to 1999 (Table 2 and 3). This means that increase in the average real income was sufficient to compensate increase in inequality from 1979 to 1999. (Real income distribution actually worsened from 1994 to 1999 due to decrease in the average real income.) The argument based only on point estimates, however, is incomplete.

Table 2 and 3 also report the asymptotic standard errors calculated from our formula. Although micro data of the National Survey of Family Income and Expenditure are not publicly available, the released grouped data contain sufficient information for our purpose. For each income decile group, they report the sample mean and the sample coefficient of variation of the annual incomes, from which we can recover the

Table 2: Sample Lorenz Curve Ordinates of the Japanese Household Annual Incomes

Decile group	1979	1984	1989	1994	1999
1	0.04	0.04	0.04	0.03	0.03
	(0.0002)	(0.0002)	(0.0002)	(0.0002)	(0.0002)
2	0.09	0.09	0.09	0.08	0.08
	(0.0003)	(0.0003)	(0.0003)	(0.0003)	(0.0003)
3	0.16	0.16	0.15	0.15	0.14
	(0.0005)	(0.0005)	(0.0005)	(0.0005)	(0.0004)
4	0.24	0.23	0.23	0.22	0.21
	(0.0006)	(0.0006)	(0.0006)	(0.0006)	(0.0005)
5	0.32	0.31	0.31	0.30	0.29
	(0.0007)	(0.0007)	(0.0008)	(0.0007)	(0.0006)
6	0.42	0.41	0.40	0.39	0.39
	(0.0008)	(0.0009)	(0.0010)	(0.0009)	(0.0007)
7	0.52	0.51	0.51	0.50	0.49
	(0.0010)	(0.0010)	(0.0011)	(0.0010)	(0.0008)
8	0.64	0.64	0.63	0.62	0.62
	(0.0011)	(0.0011)	(0.0013)	(0.0012)	(0.0008)
9	0.79	0.78	0.78	0.77	0.77
	(0.0011)	(0.0011)	(0.0014)	(0.0013)	(0.0007)
10	1.00	1.00	1.00	1.00	1.00

Note: Numbers in parentheses are asymptotic standard errors.
Source: The authors' calculation from the National Survey of Family Income and Expenditure.

Table 3: Sample GL Curve Ordinates of the Japanese Household Annual Real Incomes

Decile group	1979	1984	1989	1994	1999
1	230	230	262	260	240
	(1.3)	(1.4)	(1.5)	(1.5)	(1.4)
2	577	581	652	661	610
	(2.3)	(2.4)	(2.6)	(2.7)	(2.5)
3	978	1,001	1,120	1,152	1,059
	(3.1)	(3.5)	(3.8)	(3.9)	(3.6)
4	1,451	1,479	1,652	1,710	1,588
	(4.1)	(4.4)	(4.8)	(5.2)	(4.9)
5	1,968	2,017	2,267	2,365	2,205
	(5.0)	(5.5)	(6.1)	(6.6)	(6.3)
6	2,543	2,616	2,950	3,097	2,909
	(6.0)	(6.6)	(7.4)	(8.2)	(7.9)
7	3,180	3,298	3,738	3,943	3,713
	(7.2)	(7.9)	(9.0)	(9.8)	(9.6)
8	3,935	4,089	4,638	4,924	4,666
	(8.6)	(9.5)	(10.7)	(11.7)	(11.4)
9	4,807	5,034	5,706	6,096	5,790
	(10.5)	(11.5)	(12.9)	(14.1)	(13.9)
10	6,102	6,419	7,329	7,885	7,507
	(15.9)	(17.2)	(21.3)	(22.0)	(18.7)

Note: Thousand 2000 yen deflated by the Consumer Price Index (CPI). Numbers in parentheses are asymptotic standard errors.
Source: The authors' calculation from the National Survey of Family Income and Expenditure.

Table 4: Lorenz Dominance of Income Distributions in Japan

H_0	$t_{\min}$ test statistic	Asymptotic p-value	$\bar{\chi}^2$ test statistic	Asymptotic p-value
$L_{1979} \leq L_{1984}$	–8.56	.00	75.37	.00
$L_{1979} \leq L_{1989}$	–13.98	.00	195.33	.00
$L_{1979} \leq L_{1994}$	–24.40	.00	598.26	.00
$L_{1979} \leq L_{1999}$	–32.67	.00	1067.40	.00
$L_{1984} \leq L_{1989}$	–5.64	.00	31.79	.00
$L_{1984} \leq L_{1994}$	–15.73	.00	247.47	.00
$L_{1984} \leq L_{1999}$	–23.37	.00	553.77	.00
$L_{1989} \leq L_{1994}$	–10.84	.00	117.52	.00
$L_{1989} \leq L_{1999}$	–18.17	.00	330.20	.00
$L_{1994} \leq L_{1999}$	–7.99	.00	63.83	.00
$L_{1979} \geq L_{1984}$	2.19	1.00	0.00	1.00
$L_{1979} \geq L_{1989}$	5.05	1.00	0.00	1.00
$L_{1979} \geq L_{1994}$	8.79	1.00	0.00	1.00
$L_{1979} \geq L_{1999}$	12.52	1.00	0.00	1.00
$L_{1984} \geq L_{1989}$	0.24	.81	0.00	1.00
$L_{1984} \geq L_{1994}$	6.59	1.00	0.00	1.00
$L_{1984} \geq L_{1999}$	9.60	1.00	0.00	1.00
$L_{1989} \geq L_{1994}$	2.89	1.00	0.00	1.00
$L_{1989} \geq L_{1999}$	4.52	1.00	0.00	1.00
$L_{1994} \geq L_{1999}$	1.21	.98	0.00	1.00

Note: The asymptotic p-values are based on 10,000 random draws from the asymptotic distribution of each test statistic under the least favorable case in H_0.

sample second moment. Thus we can estimate the asymptotic variance–covariance matrix of the sample GL curve ordinates. See Appendix B for the grouped data.

With about 50,000 observations, the estimates are very accurate. It is not yet obvious, however, whether the dominance relations are statistically significant. We should formally test, for example,

$$H_0 : GL_{1999} \geq GL_{1979} \quad \text{vs.} \quad H_1 : GL_{1999} \ngeq GL_{1979},$$

where GL_{1979} and GL_{1999} are the vectors of GL curve ordinates in 1979 and 1999 respectively. We apply the $t_{\min}$ and $\bar{\chi}^2$ tests to this problem.

7.2 Testing Results

Table 4 reports the results of testing Lorenz dominance of income distributions in Japan from 1979 to 1999. For any reasonable significance level, both the $t_{\min}$ and $\bar{\chi}^2$ tests reject the null hypotheses that the Lorenz curve shifted upward, and accept the null hypotheses that it shifted downward. Thus we can statistically conclude that income inequality in Japan worsened from 1979 to 1999.

Table 5: GL Dominance of Real Income Distributions in Japan

H_0	$t_{\min}$ test statistic	Asymptotic p-value	$\bar{\chi}^2$ test statistic	Asymptotic p-value
$GL_{1979} \leq GL_{1984}$	–0.06	.75	0.003	.74
$GL_{1979} \leq GL_{1989}$	16.16	1.00	0.00	1.00
$GL_{1979} \leq GL_{1994}$	15.32	1.00	0.00	1.00
$GL_{1979} \leq GL_{1999}$	5.51	1.00	0.00	1.00
$GL_{1984} \leq GL_{1989}$	15.89	1.00	0.00	1.00
$GL_{1984} \leq GL_{1994}$	15.07	1.00	0.00	1.00
$GL_{1984} \leq GL_{1999}$	5.44	1.00	0.00	1.00
$GL_{1989} \leq GL_{1994}$	–0.69	.49	0.47	.50
$GL_{1989} \leq GL_{1999}$	–11.72	.00	141.13	.00
$GL_{1994} \leq GL_{1999}$	–17.43	.00	326.96	.00
$GL_{1979} \geq GL_{1984}$	–14.57	.00	217.91	.00
$GL_{1979} \geq GL_{1989}$	–54.09	.00	2950.10	.00
$GL_{1979} \geq GL_{1994}$	–73.31	.00	5461.80	.00
$GL_{1979} \geq GL_{1999}$	–57.29	.00	3420.50	.00
$GL_{1984} \geq GL_{1989}$	–38.93	.00	1534.50	.00
$GL_{1984} \geq GL_{1994}$	–58.37	.00	3469.40	.00
$GL_{1984} \geq GL_{1999}$	–42.87	.00	1908.80	.00
$GL_{1989} \geq GL_{1994}$	–20.41	.00	426.90	.00
$GL_{1989} \geq GL_{1999}$	–6.29	.00	39.51	.00
$GL_{1994} \geq GL_{1999}$	9.93	1.00	0.00	1.00

Note: See the note to Table 4.

Table 5 reports the results of testing GL dominance of real income distributions in Japan from 1979 to 1999. From 1979 to 1994, for any reasonable significance level, both tests accept the null hypotheses that the GL curve shifted upward, and reject the null hypotheses that it shifted downward. Thus we can statistically conclude that real income distribution in Japan improved from 1979 to 1994.

We can also statistically conclude that real income distribution in Japan worsened from 1994 to 1999. The distribution in 1999 is still better than that in 1984. Both tests reject GL dominance in both directions between 1989 and 1999. Thus we can statistically conclude that the GL curves in 1989 and 1999 cross, i.e., the two distributions cannot be ordered by GL dominance.

Notice that the $t_{\min}$ test statistics for testing GL dominance between 1979 and 1984 are negative in both directions, i.e., the sample GL curves in 1979 and 1984 cross; hence the two distribution seems incomparable. Interestingly, however, both the $t_{\min}$ and $\bar{\chi}^2$ tests clearly show GL dominance. The same is true between 1989 and 1994. This means that although GL dominance exists, the sample GL curves cross because of sampling errors.

8 DISCUSSION

This paper contributes to simplification of statistical inference for Lorenz and GL dominance in two ways. First, we give an intuitive derivation of the asymptotic distribution of a vector of sample GL curve ordinates, interpreting it as an MM estimator. Second, we propose a simple simulation-based simultaneous asymptotic one-sided t test for Lorenz and GL dominance (and for multiple inequality restrictions in general), which is consistent and asymptotically has the correct size. The results of our Monte Carlo experiments show that our $t_{\min}$ test tends to be more powerful against crossing curves, while the $\bar{\chi}^2$ test, proposed by Xu (1997) and Dardanoni and Forcina (1999), tends to be more powerful against other alternative hypotheses. Thus the two tests are complementary.

Our procedure has applications other than comparing income or wealth distributions. Since GL dominance is equivalent to the SSD, a possible application is to test the SSD of the distribution of one asset return over that of another. Since financial data typically have serial dependence, some modifications will be necessary. This may be an interesting direction for future research.

9 ACKNOWLEDGMENTS

We thank Masahito Kobayashi for useful comments.

A APPENDIX 1: PROOFS

A.1 Theorem 3

Lemma 1 *Suppose that*

1. $\operatorname{plim}_{n\to\infty} \hat{\theta}_n = \theta_0$,

2. $\{\nu_n(.)\}_{n=1}^{\infty}$ *is stochastically equicontinuous.*

Then

$$\operatorname*{plim}_{n\to\infty} \left(\nu_n\left(\hat{\theta}_n\right) - \nu_n(\theta_0)\right) = 0.$$

Proof. See Andrews (1994, pp. 2256–2257). □

Proof of Theorem 3 Since $F(.)$ is C^1 on its support, $m_0(.)$ is C^1 on Θ. Applying the mean value theorem to each component of $m_0(.)$ at $\theta = \hat{\theta}_n$,

$$m_0(\theta_0) = m_0\left(\hat{\theta}_n\right) + J_n\left(\theta_0 - \hat{\theta}_n\right),$$

where

$$J_n := \begin{bmatrix} m'_{0,1}\left(\theta^*_{n,1}\right) \\ \vdots \\ m'_{0,2k-1}\left(\theta^*_{n,2k-1}\right) \end{bmatrix},$$

and $\theta^*_{n,1}, \ldots, \theta^*_{n,2k-1}$ are the mean values. Since $m_0(\theta_0) = 0$ and J_n is nonsingular,

$$\sqrt{n}\left(\hat{\theta}_n - \theta_0\right) = J_n^{-1}\sqrt{n}m_0\left(\hat{\theta}_n\right).$$

Since $m'_0(.)$ is C^0 and $\hat{\theta}_n$ is consistent for θ_0,

$$\operatorname*{plim}_{n\to\infty} J_n = J,$$

where J is nonsingular. By the previous lemma,

$$\begin{aligned} -\sqrt{n}m_0\left(\hat{\theta}_n\right) &= \sqrt{n}\bar{m}_n\left(\hat{\theta}_n\right) - \sqrt{n}m_0\left(\hat{\theta}_n\right) - \sqrt{n}\bar{m}_n\left(\hat{\theta}_n\right) \\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^n\left(m\left(X_i;\hat{\theta}_n\right) - \mathrm{E}(m\left(X_i;\hat{\theta}_n\right)\right) + o(1) \\ &= \nu_n\left(\hat{\theta}_n\right) + o(1) \\ &= \nu_n(\theta_0) + \left(\nu_n\left(\hat{\theta}_n\right) - \nu_n(\theta_0)\right) + o(1) \\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^n(m(X_i;\theta_0) - \mathrm{E}(m(X_i;\theta_0))) + o_p(1). \end{aligned}$$

By the Lindeberg–Lévy central limit theorem,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n(m(X_i;\theta_0) - \mathrm{E}(m(X_i;\theta_0))) \to_d \mathrm{N}(0,V).$$

The result follows by Slutsky's lemma and the continuous mapping theorem. □

A.2 Theorem 4

Let $Q(.)$ be a probability measure on $(\Re, \mathcal{B}(\Re))$. Let $\mathcal{F} \subset L_2(Q)$. The *$\epsilon$-covering number* of $\mathcal{F}$ with respect to the $L_2(Q)$-norm is the smallest $N \geq 1$ such that there exist $f_1, \ldots, f_N \in \mathcal{F}$ such that for all $f \in \mathcal{F}$,

$$\min_{i\in\{1,\ldots,N\}} \|f - f_i\|_{L_2(Q)} < \epsilon.$$

Let $N_2(\epsilon; \mathcal{F}, Q(.))$ be the ϵ-covering number of $\mathcal{F}$ with respect to the $L_2(Q)$-norm. The *ϵ-entropy* of $\mathcal{F}$ with respect to the $L_2(Q)$-norm is

$$H_2(\epsilon; \mathcal{F}, Q(.)) := \ln N_2(\epsilon; \mathcal{F}, Q(.)).$$

An *envelope* for $\mathcal{F}$ is F such that for all $f \in \mathcal{F}$, $|f| \leq F$. Let $\mathcal{Q}$ be a set of probability measures on $(\Re, \mathcal{B}(\Re))$. The *uniform ϵ-entropy* of $\mathcal{F}$ on $\mathcal{Q}$ with respect to F in L_2 is

$$\sup_{Q(.) \in \mathcal{Q}} H_2 \left(\|F\|_{L_2(Q)} \epsilon; \mathcal{F}, Q(.) \right).$$

See van der Vaart (1998, p. 274) on these notions.

Definition 9 *$\mathcal{F}$ satisfies the* uniform entropy condition *with respect to F if*

$$\int_0^1 \sqrt{\sup_{Q(.) \in \mathcal{Q}} H_2 \left(\|F\|_{L_2(Q)} \epsilon; \mathcal{F}, Q(.) \right)} \mathrm{d}\epsilon < \infty,$$

where $\mathcal{Q}$ is the set of all discrete distributions on $\Re$ that take positive probabilities only on finite subsets of $\Re$.

Lemma 2 *Suppose that*

1. *$X_1, \ldots, X_n$ are iid,*
2. *$\mathcal{F} := \{f(X_1; \theta) : \theta \in \Theta\}$ satisfies the uniform entropy condition with respect to F,*
3. *$\mathrm{E}\left(F^2\right) < \infty$.*

Then the corresponding sequence of empirical processes on Θ given $(X_1, \ldots, X_n)$ is stochastically equicontinuous.

Proof. This is essentially Theorem 1 in Andrews (1994), which relies on Theorem 10.6 in Pollard (1990); see also van der Vaart and Wellner (1996, Theorem 2.5.2). We need only $\mathrm{E}\left(F^2\right) < \infty$ instead of $\mathrm{E}\left(F^{2+\delta}\right) < \infty$ for some $\delta > 0$, because it suffices for the Lindeberg condition when $X_1, \ldots, X_n$ are iid; see Davidson (1994, p. 371). $\square$

Verifying the uniform entropy condition looks awkward; hence Andrews (1994) lists various classes of random functions that satisfy the condition. The following class is relevant for our case. Let X be a k-variate random vector.

Definition 10 *$\mathcal{F}$ is a* type 1 class *if*

1. *$\mathcal{F} := \{X'\theta : \theta \in \Theta \subset \Re^k\}$ or*

2. *$\mathcal{F} := \{f(X'\theta) : \theta \in \Theta \subset \Re^k\}$, where $f(.)$ is of bounded variation.*

Proof of Theorem 4 Let $\mathcal{M} := \{m(X_1;\theta) : \theta \in \Theta\}$. By the previous lemma, it suffices to show that $\mathcal{M}$ satisfies the uniform entropy condition with an envelope that is square integrable.

Let $W := (X_1, -1, 0, \ldots)'$. Let $\Theta^* := \{1\} \times \Theta$. Let for all $\theta^* \in \Theta^*$,

$$\begin{aligned} f(W;\theta^*) &:= [W'\theta^* \leq 0] \\ &= \left[\begin{pmatrix} X_1 & -1 & 0 & \ldots \end{pmatrix}\begin{pmatrix} 1 \\ \theta \end{pmatrix} \leq 0\right] \\ &= [X_1 \leq x_1]. \end{aligned}$$

Since the indicator function is of bounded variation, $\mathcal{F} := \{f(W;\theta^*) : \theta^* \in \Theta^*\}$ is a type 1 class. By Theorem 2 in Andrews (1994), it satisfies the uniform entropy condition. An obvious envelope for $\mathcal{F}$ is $F := 1$. This argument applies to the first $k-1$ components of $m(X_1;.)$.

We have for all $\theta^* \in \Theta$,

$$f(W;\theta^*)X_1 = [X_1 \leq x_1]X_1.$$

Let $\mathcal{G} := \{X_1\}$. A singleton set trivially satisfies the uniform entropy condition. Since $\mathcal{F}$ and $\mathcal{G}$ satisfy the uniform entropy condition, $\mathcal{FG}$ satisfies the uniform entropy condition by Theorem 3 in Andrews (1994). An obvious envelope for $\mathcal{FG}$ is $F := |X_1|$. This argument applies to the last k components of $m(X_1;.)$.

Thus $\mathcal{M}$ satisfies the uniform entropy condition with an envelope

$$M := \begin{pmatrix} 1 \\ \vdots \\ 1 \\ |X_1| \\ \vdots \\ |X_1| \end{pmatrix},$$

which is square integrable. Hence $\{\nu_n(.)\}_{n=1}^{\infty}$ is stochastically equicontinuous. $\square$

A.3 Theorem 5

Proof of Theorem 5 We can write the result of Theorem 3 as

$$\sqrt{n}\left(\hat{\theta}_n - \theta\right) \to_d \mathrm{N}\left(0, \begin{bmatrix} J_{11} & O_{(k-1)\times k} \\ J_{21} & -I_k \end{bmatrix}^{-1} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} J_{11} & O_{(k-1)\times k} \\ J_{21} & -I_k \end{bmatrix}^{-1\prime}\right),$$

where

$$\begin{aligned}
J_{11} &:= \begin{bmatrix} f(x_1) & & 0 \\ & \ddots & \\ 0 & & f(x_{k-1}) \end{bmatrix}, \\
J_{21} &:= \begin{bmatrix} x_1 f(x_1) & & 0 \\ & \ddots & \\ 0 & & x_{k-1} f(x_{k-1}) \\ 0 & \dots & 0 \end{bmatrix}, \\
V_{11} &:= \begin{bmatrix} \alpha_1(1-\alpha_1) & \dots & \alpha_1(1-\alpha_{k-1}) \\ \vdots & \ddots & \vdots \\ \alpha_1(1-\alpha_{k-1}) & \dots & \alpha_{k-1}(1-\alpha_{k-1}) \end{bmatrix}, \\
V_{12} &:= \begin{bmatrix} GL_1 - \alpha_1 GL_1 & \dots & GL_1 - \alpha_1 GL_k \\ \vdots & & \vdots \\ GL_1 - \alpha_{k-1} GL_1 & \dots & GL_{k-1} - \alpha_{k-1} GL_k \end{bmatrix}, \\
V_{21} &:= \begin{bmatrix} GL_1 - GL_1\alpha_1 & \dots & GL_1 - GL_1\alpha_{k-1} \\ \vdots & & \vdots \\ GL_1 - GL_k\alpha_1 & \dots & GL_{k-1} - GL_k\alpha_{k-1} \end{bmatrix}, \\
V_{22} &:= \begin{bmatrix} \mathrm{E}\left([X_1 \le x_1]X_1^2\right) - GL_1^2 & \dots & \mathrm{E}\left([X_1 \le x_1]X_1^2\right) - GL_1 GL_k \\ \vdots & \ddots & \vdots \\ \mathrm{E}\left([X_1 \le x_1]X_1^2\right) - GL_k GL_1 & \dots & \mathrm{E}\left([X_1 \le x_k]X_1^2\right) - GL_k^2 \end{bmatrix}.
\end{aligned}$$

Notice that

$$\begin{bmatrix} J_{11} & O_{(k-1)\times k} \\ J_{21} & -I_k \end{bmatrix}^{-1} = \begin{bmatrix} J_{11}^{-1} & O_{(k-1)\times k} \\ J_{21}J_{11}^{-1} & -I_k \end{bmatrix},$$

where

$$J_{21}J_{11}^{-1} = \begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_{k-1} \\ 0 & \dots & 0 \end{bmatrix}.$$

Thus

$$\begin{aligned}
\Sigma &= J_{21}J_{11}^{-1}V_{11}J_{11}^{-1}J_{21}' - J_{21}J_{11}^{-1}V_{12} - V_{21}J_{11}^{-1}J_{21}' + V_{22} \\
&= A - B - C + V_{22},
\end{aligned}$$

where

$$A := J_{21}J_{11}^{-1}V_{11}J_{11}^{-1}J_{21}'$$

$$
= \begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_{k-1} \\ 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \alpha_1(1-\alpha_1) & \dots & \alpha_1(1-\alpha_{k-1}) \\ \vdots & \ddots & \vdots \\ \alpha_1(1-\alpha_{k-1}) & \dots & \alpha_{k-1}(1-\alpha_{k-1}) \end{bmatrix}
$$

$$
\begin{bmatrix} x_1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & x_{k-1} & 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} x_1\alpha_1(1-\alpha_1)x_1 & \dots & x_1\alpha_1(1-\alpha_{k-1})x_{k-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ x_{k-1}\alpha_1(1-\alpha_{k-1})x_1 & \dots & x_{k-1}\alpha_{k-1}(1-\alpha_{k-1})x_{k-1} & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix}
$$

$$
B := J_{21}J_{11}^{-1}V_{12}
$$

$$
= \begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_{k-1} \\ 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} GL_1 - \alpha_1 GL_1 & \dots & GL_1 - \alpha_1 GL_k \\ \vdots & & \vdots \\ GL_1 - \alpha_{k-1} GL_1 & \dots & GL_{k-1} - \alpha_{k-1} GL_k \end{bmatrix}
$$

$$
= \begin{bmatrix} x_1(GL_1 - \alpha_1 GL_1) & \dots & x_1(GL_1 - \alpha_1 GL_k) \\ \vdots & & \vdots \\ x_{k-1}(GL_1 - \alpha_{k-1} GL_1) & \dots & x_{k-1}(GL_{k-1} - \alpha_{k-1} GL_k) \\ 0 & \dots & 0 \end{bmatrix},
$$

$$
C := V_{21}J_{11}^{-1}J_{21}
$$

$$
= \begin{bmatrix} GL_1 - GL_1\alpha_1 & \dots & GL_1 - GL_1\alpha_{k-1} \\ \vdots & & \vdots \\ GL_1 - GL_k\alpha_1 & \dots & GL_{k-1} - GL_k\alpha_{k-1} \end{bmatrix} \begin{bmatrix} x_1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & x_{k-1} & 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} (GL_1 - GL_1\alpha_1)x_1 & \dots & (GL_1 - GL_1\alpha_{k-1})x_1 & 0 \\ \vdots & & \vdots & \vdots \\ (GL_1 - GL_k\alpha_1)x_1 & \dots & (GL_{k-1} - GL_{k-1}\alpha_{k-1})x_{k-1} & 0 \end{bmatrix}.
$$

□

A.4 Theorem 6

Proof of Theorem 6

1. ($\theta_{\min}$ test) Let for all $\alpha \in [0,1]$, c_α be such that

$$
\lim_{n\to\infty} \Pr\left[\sqrt{n}\hat{\theta}_{n,\min} \le c_\alpha | \theta_0 = 0\right] = \alpha.
$$

   We want to show that for all $\alpha \in [0,1]$,

$$
\lim_{n\to\infty} \sup_{\theta \ge 0} \Pr\left[\sqrt{n}\hat{\theta}_{n,\min} \le c_\alpha | \theta_0 = \theta\right] = \alpha.
$$

   Let $1_k := (1,\dots,1)'$. Since $\hat{\theta}_n$ is asymptotically normal, for all $\alpha \in [0,1]$, for all $\theta \ge 0$,

$$
\lim_{n\to\infty} \Pr\left[\sqrt{n}\hat{\theta}_n > c_\alpha 1_k | \theta_0 = \theta\right] \ge \lim_{n\to\infty} \Pr\left[\sqrt{n}\hat{\theta}_n > c_\alpha 1_k | \theta_0 = 0\right],
$$

or

$$\lim_{n\to\infty} \Pr\left[\sqrt{n}\hat{\theta}_{n,\min} > c_\alpha | \theta_0 = \theta\right] \geq \lim_{n\to\infty} \Pr\left[\sqrt{n}\hat{\theta}_{n,\min} > c_\alpha | \theta_0 = 0\right].$$

Hence

$$\begin{aligned}
\lim_{n\to\infty} \Pr\left[\sqrt{n}\hat{\theta}_{n,\min} \leq c_\alpha | \theta_0 = \theta\right] &= 1 - \lim_{n\to\infty} \Pr\left[\sqrt{n}\hat{\theta}_{n,\min} > c_\alpha | \theta_0 = \theta\right] \\
&\leq 1 - \lim_{n\to\infty} \Pr\left[\sqrt{n}\hat{\theta}_{n,\min} > c_\alpha | \theta_0 = 0\right] \\
&= \lim_{n\to\infty} \Pr\left[\sqrt{n}\hat{\theta}_{n,\min} \leq c_\alpha | \theta_0 = 0\right] \\
&= \alpha.
\end{aligned}$$

2. ($t_{\min}$ test) Repeat the same argument for $t_{n,\min}$.

□

A.5 Theorem 7

Proof of Theorem 7

1. ($\theta_{\min}$ test) Let for all $\alpha \in [0,1]$, c_α be such that

$$\lim_{n\to\infty} \Pr\left[\sqrt{n}\hat{\theta}_{n,\min} \leq c_\alpha | \theta_0 = 0\right] = \alpha.$$

We want to show that for all $\theta \ngeq 0$,

$$\lim_{n\to\infty} \Pr\left[\sqrt{n}\hat{\theta}_{n,\min} \leq c_\alpha | \theta_0 = \theta\right] = 1.$$

Assume without loss of generality that $\theta_{\min} = \theta_{0,1} < 0$. Then

$$\begin{aligned}
\operatorname*{plim}_{n\to\infty} \sqrt{n}\hat{\theta}_{n,\min} &= \operatorname*{plim}_{n\to\infty} \min\left\{\sqrt{n}\hat{\theta}_{n,1}, \ldots, \sqrt{n}\hat{\theta}_{n,k}\right\} \\
&\leq \operatorname*{plim}_{n\to\infty} \sqrt{n}\hat{\theta}_{n,1} \\
&= \operatorname*{plim}_{n\to\infty} \sqrt{n}\left(\hat{\theta}_{n,1} - \theta_{0,1}\right) + \lim_{n\to\infty} \sqrt{n}\theta_{0,1} \\
&= -\infty.
\end{aligned}$$

2. ($t_{\min}$ test) Let for all $\alpha \in [0,1]$, c_α be such that

$$\lim_{n\to\infty} \Pr[t_{n,\min} \leq c_\alpha | \theta_0 = 0] = \alpha.$$

We want to show that for all $\theta \ngeq 0$,

$$\lim_{n\to\infty} \Pr[t_{n,\min} \leq c_\alpha | \theta_0 = \theta] = 1.$$

Assume without loss of generality that $\theta_{\min} = \theta_{0,1} < 0$. Then

$$\begin{aligned}
\operatorname*{plim}_{n\to\infty} t_{n,\min} &= \operatorname*{plim}_{n\to\infty} \min\{t_{n,1}, \ldots, t_{n,k}\} \\
&= \operatorname*{plim}_{n\to\infty} \min\left\{\frac{\sqrt{n}\hat{\theta}_{n,1}}{\hat{\sigma}_{n,1}}, \ldots, \frac{\sqrt{n}\hat{\theta}_{n,k}}{\hat{\sigma}_{n,k}}\right\} \\
&\leq \operatorname*{plim}_{n\to\infty} \frac{\sqrt{n}\hat{\theta}_{n,1}}{\hat{\sigma}_{n,1}} \\
&= \operatorname*{plim}_{n\to\infty} \frac{\sqrt{n}\left(\hat{\theta}_{n,1} - \theta_{0,1}\right)}{\hat{\sigma}_{n,1}} + \operatorname*{plim}_{n\to\infty} \frac{\sqrt{n}\theta_{0,1}}{\hat{\sigma}_{n,1}} \\
&= -\infty.
\end{aligned}$$

□

B APPENDIX 2: DATA

Table 6 shows the grouped data of the Japanese household annual incomes used in the application. The data are from the National Survey of Family Income and Expenditure published by the Statistics Bureau, the Ministry of Public Management, Home Affairs, Posts and Telecommunications. Although they do not use simple random sampling, they make appropriate adjustments for the grouped data; thus we can simply apply our formula for the asymptotic variance–covariance matrix of a vector of sample GL curve ordinates to this data. Note that the data are before-tax and exclude one-person households. Table 7 shows the CPI in Japan used to deflate nominal incomes to real incomes.

Table 6: Grouped Data of the Japanese Household Annual Incomes

Year	Decile group	Decile (thousand yen)	Frequency	Mean (thousand yen)	Coefficient of variation
1979	1	2,050	5,046	1,550	26.7
	2	2,600	4,999	2,364	6.4
	3	3,000	4,814	2,835	4.6
	4	3,440	5,013	3,212	4.0
	5	3,800	4,869	3,613	3.0
	6	4,250	4,855	4,032	3.0
	7	4,820	4,788	4,534	3.5
	8	5,600	4,972	5,168	4.1
	9	6,990	4,806	6,176	6.4
	10	—	4,610	9,564	39.4
1984	1	2,500	5,259	1,841	27.1
	2	3,200	5,148	2,875	7.3
	3	3,800	5,049	3,511	4.6
	4	4,310	4,966	4,055	3.7
	5	4,860	4,954	4,583	3.4
	6	5,470	4,909	5,141	3.3
	7	6,200	4,941	5,816	3.7
	8	7,240	4,981	6,700	4.6
	9	9,000	4,963	8,030	6.3
	10	—	4,784	12,203	42.0
1989	1	2,940	5,891	2,174	27.8
	2	3,760	5,697	3,352	7.2
	3	4,500	5,558	4,123	4.9
	4	5,100	5,408	4,809	4.0
	5	5,850	5,505	5,474	3.8
	6	6,600	5,378	6,208	3.5
	7	7,560	5,458	7,068	3.9
	8	8,850	5,400	8,159	4.5
	9	10,980	5,343	9,783	6.0
	10	—	5,163	15,380	56.0
1994	1	3,330	5,723	2,464	28.1
	2	4,350	5,619	3,861	7.5
	3	5,200	5,557	4,786	5.0
	4	6,000	5,395	5,606	4.4
	5	6,900	5,522	6,430	4.2
	6	7,860	5,394	7,348	3.9
	7	9,000	5,472	8,378	3.9
	8	10,500	5,482	9,696	4.5
	9	13,030	5,442	11,668	6.3
	10	—	5,346	18,136	47.9
1999	1	3,210	5,576	2,351	24.4
	2	4,130	5,461	3,700	6.1
	3	4,960	5,395	4,538	4.4
	4	5,780	5,405	5,344	3.7
	5	6,640	5,434	6,200	3.4
	6	7,630	5,384	7,139	3.3
	7	8,800	5,354	8,190	3.6
	8	10,290	5,471	9,515	3.8
	9	12,940	5,368	11,428	5.8
	10	—	5,357	17,495	22.5

Note: The data exclude one-person households.
Source: National Survey of Family Income and Expenditure.

Table 7: CPI in Japan (2000=100)

Year	CPI
1979	69.8
1984	84.4
1989	89.3
1994	98.6
1999	100.7

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