ECONOMETRICS: HOMEWORK 6

DUE ON DECEMBER 12, 2014

Instructions: Answer ALL questions. Consult books when necessary. Work together with your classmates, but turn in separately. Type up your answers using LaTeX.

(1) Let $(\boldsymbol{y}, \boldsymbol{X})$ be a sample. Assume a GLRM for $\boldsymbol{y}$ given $\boldsymbol{X}$ s.th.

$$\begin{aligned}\mathrm{E}(\boldsymbol{y}|\boldsymbol{X}) &= \boldsymbol{X}\boldsymbol{\beta}\\ \operatorname{var}(\boldsymbol{y}|\boldsymbol{X}) &= \boldsymbol{\Sigma}(\boldsymbol{X})\end{aligned}$$

Let $\boldsymbol{b}$ be the OLS estimator of $\boldsymbol{\beta}$.

(a) Show that $\boldsymbol{b}$ is unbiased for $\boldsymbol{\beta}$.

(b) Derive the (unconditional) variance of $\boldsymbol{b}$.

(2) Solve the GLS problem

$$\begin{aligned}\min_{\boldsymbol{\beta}} \quad & (\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})\\ \text{and} \quad & \boldsymbol{\beta} \in \mathbb{R}^k\end{aligned}$$

(3) Let $(\boldsymbol{y}, \boldsymbol{X})$ be a random sample of size n. Assume an LRM with conditional heteroskedasticity for y_i given $\boldsymbol{x}_i$ s.th. for $i = 1, \ldots, n$,

$$\begin{aligned}\mathrm{E}(y_i|\boldsymbol{x}_i) &= \boldsymbol{x}_i'\boldsymbol{\beta}\\ \operatorname{var}(y_i|\boldsymbol{x}_i) &= \sigma^2(\boldsymbol{x}_i)\end{aligned}$$

Assume that we know $\sigma^2(.)$. Let $\boldsymbol{b}_{G,n}$ be the infeasible GLS estimator of $\boldsymbol{\beta}$.

(a) Show that $\boldsymbol{b}_{G,n}$ is consistent for $\boldsymbol{\beta}$.

(b) Show that

$$\sqrt{n}(\boldsymbol{b}_{G,n} - \boldsymbol{\beta}) \xrightarrow{d} \mathrm{N}\left(\boldsymbol{0}, \mathrm{E}\left(\frac{\boldsymbol{x}_1\boldsymbol{x}_1'}{\sigma^2(\boldsymbol{x}_1)}\right)^{-1}\right)$$

(4) Let $(\boldsymbol{y}, \boldsymbol{X})$ be a sample. Assume a GLRM for $\boldsymbol{y}$ given $\boldsymbol{X}$ s.th.

$$\begin{aligned}\mathrm{E}(\boldsymbol{y}|\boldsymbol{X}) &= \boldsymbol{X}\boldsymbol{\beta}\\ \operatorname{var}(\boldsymbol{y}|\boldsymbol{X}) &= \boldsymbol{\Sigma}(\boldsymbol{X})\end{aligned}$$

Let $\boldsymbol{b}$ be the OLS estimator and $\boldsymbol{b}_G$ be the infeasible GLS estimator of $\boldsymbol{\beta}$ respectively. Then

$$\begin{aligned}\operatorname{var}(\boldsymbol{b}|\boldsymbol{X}) &= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{\Sigma}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\\ \operatorname{var}(\boldsymbol{b}_G|\boldsymbol{X}) &= \left(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X}\right)^{-1}\end{aligned}$$

Show directly, i.e., without using the G–M theorem, that

$$\operatorname{var}(\boldsymbol{b}_G|\boldsymbol{X}) \leq \operatorname{var}(\boldsymbol{b}|\boldsymbol{X})$$

(Hint: $\boldsymbol{A} - \boldsymbol{B}$ is p.s.d. iff $\boldsymbol{B}^{-1} - \boldsymbol{A}^{-1}$ is p.s.d.)

(5) Let $(\boldsymbol{y}, \boldsymbol{X}, \boldsymbol{Z})$ be a $(1+k+l)$-variate random sample of size n, where $l \geq k$. Assume a linear model for y_i s.th. for $i = 1, \ldots, n$,

$$y_i = \boldsymbol{x}_i'\boldsymbol{\beta} + u_i$$

Assume that $\boldsymbol{z}_i$ is an IV vector for estimating $\boldsymbol{\beta}$.

(a) Define a GMM estimator of $\boldsymbol{\beta}$. Call it $\hat{\boldsymbol{\beta}}_n$.

(b) Assume that the weight matrix $\boldsymbol{W}_n$ is symmetric and p.d. Show that

$$\hat{\boldsymbol{\beta}}_n = (\boldsymbol{X}'\boldsymbol{Z}\boldsymbol{W}_n\boldsymbol{Z}'X)^{-1}\boldsymbol{X}'\boldsymbol{Z}\boldsymbol{W}_n\boldsymbol{Z}'\boldsymbol{y}$$

(c) Show that $\hat{\boldsymbol{\beta}}_n$ is consistent for $\boldsymbol{\beta}$.

(d) Derive the asymptotic distribution of $\hat{\boldsymbol{\beta}}_n$.

(e) The two-stage least squares (2SLS) estimator of $\boldsymbol{\beta}$ is

$$b_{2\text{SLS},n} = \left[\boldsymbol{X}'\boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{X}\right]^{-1}\boldsymbol{X}'\boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{y}$$

Show that this is an optimal GMM estimator of $\boldsymbol{\beta}$ if $\text{var}(u_i|\boldsymbol{z}_i) = \sigma^2$.

(f) Let $\hat{\boldsymbol{X}}$ be the projection of $\boldsymbol{X}$ on $\boldsymbol{Z}$. Show that

$$\boldsymbol{b}_{2\text{SLS},n} = \left(\hat{\boldsymbol{X}}'\hat{\boldsymbol{X}}\right)^{-1}\hat{\boldsymbol{X}}\boldsymbol{y}$$

(This is where the name comes from.)