LECTURE 27: SIMULTANEOUS EQUATIONS MODELS

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1. Simultaneous Equations Models

1.1. **Economic Models.** Consider a market in perfect competition. Let

- Q_d be the aggregate demand,
- Q_s be the aggregate supply,
- P be the unit price of the commodity,
- Y be the aggregate real income of the buyers,
- C be the average cost of production.

An economic theory may imply that

$$\ln Q_d = \alpha_0 + \alpha_1 \ln P + \alpha_2 \ln Y + u$$
$$\ln Q_s = \beta_0 + \beta_1 \ln P + \beta_2 \ln C + v$$

where u and v capture possible omitted variables and measurement errors. Assume that the market is in a competitive equilibrium. Then the market clearing condition determines the observed equilibrium price P^* and quantity Q^* *simultaneously* given Y and C.

Definition 1. A variable is *exogenous* if it is determined outside the system.

Definition 2. A variable is *endogenous* if it is determined inside the system given exogenous variables.

1.2. **Linear Simultaneous Equations Models.**

1.2.1. *Structural Form.* Let $(\boldsymbol{Y}, \boldsymbol{X})$ be a $(G+K)$-variate random sample of size n. Assume a linear simultaneous equations model (SEM) for $\boldsymbol{y}_i$ given $\boldsymbol{x}_i$ s.th. for $i = 1, \ldots, n$,

$$\boldsymbol{\Gamma} \boldsymbol{y}_i = \boldsymbol{B} \boldsymbol{x}_i + \boldsymbol{u}_i$$
$$\mathrm{E}(\boldsymbol{u}_i | \boldsymbol{x}_i) = \boldsymbol{0}$$
$$\mathrm{var}(\boldsymbol{u}_i | \boldsymbol{x}_i) = \boldsymbol{\Sigma}$$

Assume w.l.o.g. that $\boldsymbol{\Gamma}$ is nonsingular (otherwise we have a redundant equation).

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1.2.2. *Reduced Form.* The reduced form is for $i = 1, \dots, n$,

$$\begin{aligned} \boldsymbol{y}_i &= \boldsymbol{\Pi} \boldsymbol{x}_i + \boldsymbol{v}_i \\ \mathrm{E}(\boldsymbol{v}_i | \boldsymbol{x}_i) &= \boldsymbol{0} \\ \mathrm{var}(\boldsymbol{v}_i | \boldsymbol{x}_i) &= \boldsymbol{\Omega} \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{\Pi} &:= \boldsymbol{\Gamma}^{-1} \boldsymbol{B} \\ \boldsymbol{\Omega} &:= \boldsymbol{\Gamma}^{-1} \boldsymbol{\Sigma} \boldsymbol{\Gamma}^{-1\prime} \end{aligned}$$

This is a multivariate LRM for $\boldsymbol{y}_i$ given $\boldsymbol{x}_i$.

1.3. **Simultaneous Equations Bias.** Assume a linear SEM for $\boldsymbol{y}_i$ given $\boldsymbol{x}_i$ s.th. for $i = 1, \dots, n$,

$$\begin{aligned} y_{i,1} &= -\gamma_{1,2} y_{i,2} + \boldsymbol{x}_i' \boldsymbol{\beta}_1 + u_{i,1} \\ y_{i,2} &= -\gamma_{2,1} y_{i,1} + \boldsymbol{x}_i' \boldsymbol{\beta}_2 + u_{i,2} \\ \mathrm{E}(\boldsymbol{u}_i | \boldsymbol{x}_i) &= \boldsymbol{0} \\ \mathrm{var}(\boldsymbol{u}_i | \boldsymbol{x}_i) &= \boldsymbol{\Sigma} \end{aligned}$$

Suppose that we apply OLS to the first equation. We have for $i = 1, \dots, n$,

$$\begin{aligned} \mathrm{E}(y_{i,2} u_{i,1}) &= \mathrm{E}((-\gamma_{2,1} y_{i,1} + \boldsymbol{x}_i' \boldsymbol{\beta}_2 + u_{i,2}) u_{i,1}) \\ &= -\gamma_{2,1} \mathrm{E}(y_{i,1} u_{i,1}) + \sigma_{1,2} \\ &= -\gamma_{2,1} \mathrm{E}((-\gamma_{1,2} y_{i,2} + \boldsymbol{x}_i' \boldsymbol{\beta}_1 + u_{i.1}) u_{i,1}) + \sigma_{1,2} \\ &= \gamma_{2,1} \gamma_{1,2} \mathrm{E}(y_{i,2} u_{i,1}) - \gamma_{2,1} \sigma_{1,1} + \sigma_{1,2} \\ &= \frac{-\gamma_{2,1} \sigma_{1,1} + \sigma_{1,2}}{1 - \gamma_{2,1} \gamma_{1,2}} \end{aligned}$$

So the OLS estimator is inconsistent unless $\gamma_{2,1} = \sigma_{1,2} = 0$.

2. Identification

Let $\gamma_{1,1} := 1$ and consider the first equation. Let

- $\boldsymbol{y}_{i,2}$ be the other G_1 endogenous variables in the first structural equation,
- $\boldsymbol{x}_{i,1}$ be the K_1 exogenous variables in the first structural equation,
- $\boldsymbol{x}_{i,2}$ be the other K_2 exogenous variables.

We have for $i = 1, \dots, n$,

$$\begin{bmatrix} 1 & \boldsymbol{\gamma}' \\ \boldsymbol{0}_{G_1} & \boldsymbol{\Gamma}_2 \end{bmatrix} \begin{pmatrix} y_{i,1} \\ \boldsymbol{y}_{i,2} \end{pmatrix} = \begin{bmatrix} \boldsymbol{\beta}' & \boldsymbol{0}_{K_2}' \\ \boldsymbol{0}_{K_1} & \boldsymbol{B}_2 \end{bmatrix} \begin{pmatrix} \boldsymbol{x}_{i,1} \\ \boldsymbol{x}_{i,2} \end{pmatrix} + \begin{pmatrix} u_{i,1} \\ \boldsymbol{u}_{i,2} \end{pmatrix}$$

The first structural equation is for $i = 1, \dots, n$,

$$y_{i,1} + \boldsymbol{\gamma}' \boldsymbol{y}_{i,2} = \boldsymbol{\beta}' \boldsymbol{x}_{i,1} + u_{i,1}$$

The reduced form is for $i = 1, \dots, n$,

$$\begin{pmatrix} y_{i,1} \\ \boldsymbol{y}_{i,2} \end{pmatrix} = \begin{bmatrix} \boldsymbol{\pi}_{1,1}' & \boldsymbol{\pi}_{1,2}' \\ \boldsymbol{\Pi}_{2,1} & \boldsymbol{\Pi}_{2,2} \end{bmatrix} \begin{pmatrix} \boldsymbol{x}_{i,1} \\ \boldsymbol{x}_{i,2} \end{pmatrix} + \begin{pmatrix} v_{i,1} \\ \boldsymbol{v}_{i,2} \end{pmatrix}$$

Premultiplying it by $(1, \boldsymbol{\gamma}')$,

$$y_{i,1} + \boldsymbol{\gamma}' \boldsymbol{y}_{i,2} = \begin{pmatrix} \boldsymbol{\pi}_{1,1}' + \boldsymbol{\gamma}' \boldsymbol{\Pi}_{2,1} & \boldsymbol{\pi}_{1,2}' + \boldsymbol{\gamma}' \boldsymbol{\Pi}_{2,2} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_{i,1} \\ \boldsymbol{x}_{i,2} \end{pmatrix} + v_{i,1} + \boldsymbol{\gamma}' \boldsymbol{v}_{i,2}$$

Comparing it with the first structural equation,

$$\begin{aligned} \boldsymbol{\pi}_{1,1}' + \boldsymbol{\gamma}' \boldsymbol{\Pi}_{2,1} &= \boldsymbol{\beta}' \\ \boldsymbol{\pi}_{1,2}' + \boldsymbol{\gamma}' \boldsymbol{\Pi}_{2,2} &= \boldsymbol{0}' \end{aligned}$$

or

$$\boldsymbol{\pi}_{1,1} = -\boldsymbol{\Pi}_{2,1}'\boldsymbol{\gamma} + \boldsymbol{\beta}$$
$$\boldsymbol{\pi}_{1,2} = -\boldsymbol{\Pi}_{2,2}'\boldsymbol{\gamma}$$

Note that $\boldsymbol{\Pi}_{2,2}'$ is $K_2 \times G_1$.

Definition 3. The *rank condition* for identification of $(\boldsymbol{\gamma}', \boldsymbol{\beta}')'$ is

$$\text{rk}(\boldsymbol{\Pi}_{2,2}) \geq G_1$$

Remark 1. This is a necessary and sufficient condition for identification. Without knowing $\boldsymbol{\Pi}_{2,2}$, however, one cannot verify the rank condition.

Definition 4. The *order condition* for identification of $(\boldsymbol{\gamma}', \boldsymbol{\beta}')'$ is

$$K_2 \geq G_1$$

3. System Estimation

3.1. **FIML Estimator.** Assume a linear SEM for $\boldsymbol{y}_i$ given $\boldsymbol{x}_i$ s.th. for $i = 1, \ldots, n$,

$$\boldsymbol{\Gamma}\boldsymbol{y}_i | \boldsymbol{x}_i \sim \text{IN}(\boldsymbol{B}\boldsymbol{x}_i, \boldsymbol{\Sigma})$$

The reduced form is for $i = 1, \ldots, n$,

$$\boldsymbol{y}_i | \boldsymbol{x}_i \sim \text{IN}(\boldsymbol{\Pi}\boldsymbol{x}_i, \boldsymbol{\Omega})$$

where

$$\boldsymbol{\Pi} := \boldsymbol{\Gamma}^{-1}\boldsymbol{B}$$
$$\boldsymbol{\Omega} := \boldsymbol{\Gamma}^{-1}\boldsymbol{\Sigma}\boldsymbol{\Gamma}^{-1\prime}$$

Let

- $\boldsymbol{\gamma}$ be a vector of the unknown parameters in $\boldsymbol{\Gamma}$,
- $\boldsymbol{\beta}$ be a vector of the unknown parameters in $\boldsymbol{B}$,
- $\boldsymbol{\theta} := (\boldsymbol{\gamma}', \boldsymbol{\beta}', \text{vech}(\boldsymbol{\Sigma})')'$.

Write $\boldsymbol{\Pi} = \boldsymbol{\Pi}(\boldsymbol{\theta})$ and $\boldsymbol{\Omega} = \boldsymbol{\Omega}(\boldsymbol{\theta})$. The likelihood function of $\boldsymbol{\theta}$ given $(\boldsymbol{Y}, \boldsymbol{X})$ is

$$L(\boldsymbol{\theta}; \boldsymbol{Y}, \boldsymbol{X}) = \prod_{i=1}^{n} (2\pi)^{-G/2} \det(\boldsymbol{\Omega}(\boldsymbol{\theta}))^{-1/2} \exp\left(-\frac{1}{2}(\boldsymbol{y}_i - \boldsymbol{\Pi}(\boldsymbol{\theta})\boldsymbol{x}_i)'\boldsymbol{\Omega}(\boldsymbol{\theta})^{-1}(\boldsymbol{y}_i - \boldsymbol{\Pi}(\boldsymbol{\theta})\boldsymbol{x}_i)\right)$$

The log-likelihood function of $\boldsymbol{\theta}$ given $(\boldsymbol{Y}, \boldsymbol{X})$ is

$$\ell(\boldsymbol{\theta}; \boldsymbol{Y}, \boldsymbol{X}) = -\frac{nG}{2}\ln 2\pi - \frac{n}{2}\ln\det(\boldsymbol{\Omega}(\boldsymbol{\theta})) - \frac{1}{2}\sum_{i=1}^{n}(\boldsymbol{y}_i - \boldsymbol{\Pi}(\boldsymbol{\theta})\boldsymbol{x}_i)'\boldsymbol{\Omega}(\boldsymbol{\theta})^{-1}(\boldsymbol{y}_i - \boldsymbol{\Pi}(\boldsymbol{\theta})\boldsymbol{x}_i)$$

3.2. **GMM Estimator.** By the LIE,

$$\text{E}(\boldsymbol{u}_1 \otimes \boldsymbol{x}_1) = \boldsymbol{0}$$

or

$$\text{E}((\boldsymbol{\Gamma}\boldsymbol{y}_1 - \boldsymbol{B}\boldsymbol{x}_1) \otimes \boldsymbol{x}_1) = \boldsymbol{0}$$

Write $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}(\boldsymbol{\gamma})$ and $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{\beta})$. A GMM estimator of $(\boldsymbol{\gamma}', \boldsymbol{\beta}')'$ solves

$$\min_{\boldsymbol{\gamma},\boldsymbol{\beta}} \left(\frac{1}{n}\sum_{i=1}^{n}(\boldsymbol{\Gamma}(\boldsymbol{\gamma})\boldsymbol{y}_i - \boldsymbol{B}(\boldsymbol{\beta})\boldsymbol{x}_i) \otimes \boldsymbol{x}_i\right)' \boldsymbol{W}_n \left(\frac{1}{n}\sum_{i=1}^{n}(\boldsymbol{\Gamma}(\boldsymbol{\gamma})\boldsymbol{y}_i - \boldsymbol{B}(\boldsymbol{\beta})\boldsymbol{x}_i) \otimes \boldsymbol{x}_i\right)$$

where $\boldsymbol{W}_n$ is a p.d. weight matrix.