ECONOMETRICS: PROBLEM SET 6

ID#_____NAME_____

Instructions: Answer ALL questions. Consult books or your classmates if necessary. Turn in your work in my mailbox before leaving school today.

(1) Let X be a sample from $f(.;\theta)$, where $\theta \in \Theta$. Let $t(X)$ be an estimator of θ.

(a) Define an unbiased estimator of θ.

(b) Define a UMV unbiased estimator of θ.

(c) Show that a UMV unbiased estimator minimizes the MSE uniformly on Θ among unbiased estimators.

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(2) Let X be a random sample of size n from $\mathrm{N}\left(\mu, \sigma^2\right)$. Let $\theta := \left(\mu, \sigma^2\right)$.

(a) Write the joint pdf of X.

(b) Write the log-likelihood function of θ given $X = x$.

(c) Derive the score function of θ given $X = x$.

(d) Derive Fisher's information about θ in X.

(e) Derive the CRLB for the variance of an unbiased estimator of θ.

Suggested answers:

(1) (a) $t(X)$ is unbiased for θ if $\forall\theta \in \Theta$,

$$\mathrm{E}_\theta(t(X)) = \theta$$

(b) $t^*(X)$ is uniformly minimum variance (UMV) unbiased for $\boldsymbol{\theta}$ if

(i) it is unbiased,

(ii) $\forall t(X)$ that is unbiased, $\forall\theta \in \Theta$,

$$\mathrm{var}_\theta(t^*(X)) \leq \mathrm{var}_\theta(t(X))$$

(c) We have

$$\begin{aligned}
&\mathrm{MSE}(\theta; t(.)) \\
&:= \mathrm{E}_\theta((t(X)-\theta)(t(X)-\theta)') \\
&= \mathrm{E}_\theta([(t(X)-\mathrm{E}_\theta(t(X))) + (\mathrm{E}_\theta(t(X))-\theta)] \\
&\quad [(t(X)-\mathrm{E}_\theta(t(X))) + (\mathrm{E}_\theta(t(X))-\theta)]') \\
&= \mathrm{E}_\theta([(t(X)-\mathrm{E}_\theta(t(X))) + b(\theta)][(t(X)-\mathrm{E}_\theta(t(X))) + b(\theta)]') \\
&= \mathrm{E}_\theta((t(X)-\mathrm{E}_\theta(t(X)))(t(X)-\mathrm{E}_\theta(t(X)))') \\
&\quad + \mathrm{E}_\theta(t(X)-\mathrm{E}_\theta(t(X)))b(\theta)' + b(\theta)\,\mathrm{E}_\theta(t(X)-\mathrm{E}_\theta(t(X))) + b(\theta)b(\theta)' \\
&= \mathrm{var}_\theta(t(X)) + b(\theta)b(\theta)'
\end{aligned}$$

The result follows because the second term is 0 for unbiased estimators.

(2) (a) The joint pdf of X is $\forall x \in \mathbb{R}^n$,

$$f_X(x;\theta) = (2\pi)^{-n/2}\left(\sigma^2\right)^{-n/2}\exp\left(-\frac{\sum_{i=1}^n (x_i-\mu)^2}{2\sigma^2}\right)$$

(b) The log-likelihood function of θ given $X = x$ is $\forall\theta \in \Theta$,

$$\ell(\theta; x) = -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2$$

(c) Differentiating w.r.t. θ, $\forall\theta \in \Theta$,

$$\ell_\mu(\theta; x) = \frac{1}{\sigma^2}\sum_{i=1}^n (x_i-\mu),$$

$$\ell_{\sigma^2}(\theta; x) = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^n (x_i-\mu)^2$$

(d) Differentiating once again, $\forall\theta \in \Theta$,

$$\ell_{\mu\mu'}(\theta; x) = -\frac{n}{\sigma^2}$$

$$\ell_{\mu\sigma^2}(\theta; x) = -\frac{1}{\sigma^4}\sum_{i=1}^n (x_i-\mu)$$

$$\ell_{\sigma^2\sigma^2}(\theta; x) = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}\sum_{i=1}^n (x_i-\mu)^2$$

So $\forall\theta \in \Theta$,

$$\boldsymbol{\mathcal{I}}(\theta) = \begin{bmatrix} n/\sigma^2 & 0 \\ 0 & n/\left(2\sigma^4\right) \end{bmatrix}$$

(e) The CRLB is

$$\boldsymbol{\mathcal{I}}(\theta)^{-1} = \begin{bmatrix} \sigma^2/n & 0 \\ 0 & 2\sigma^4/n \end{bmatrix}$$